

§ 1.3. Method of variation of parameters:

We shall now explain a method for finding the complete primitive of a linear eqn whose C.F. is known.

This method is applied if the C.F. is known and it is difficult to find P.I. with the methods already known to us.

Let us consider the eqn

$$y'' + py' + qy = R \quad \text{--- (1)}$$

Let C.F. is $c_1 u + c_2 v$ where c_1 and c_2 are constants.

$$\text{Then } u'' + pu' + qu = 0 \text{ and } v'' + pv' + qv = 0 \quad \text{--- (2)}$$

Let the solⁿ of (1) be

$$y = Au + Bv \quad \text{--- (3)}$$

where A and B are functions of x , known as parameters.

$$(3) \Rightarrow y' = A'u + B'v + Au' + Bv'$$

Let A and B satisfy additional condition

$$A'u + B'v = 0 \quad \text{--- (4)}$$

$$\text{Then } y' = Au' + Bv'$$

$$\Rightarrow y'' = A'u' + B'v' + Au'' + Bv''$$

Substituting these in (1) we get —

$$A(u'' + pu' + qu) + B(v'' + pv' + qv) + A'u' + B'v' = R$$

$$\Rightarrow A'U + B'v = R \quad \text{--- (5)} \quad \text{By (2)}$$

Solving (4) and (5) for A' and B' we get -

$$\frac{A'}{-vR-0} = \frac{-B'}{-UR-0} = \frac{1}{uv'-u'v}$$

$$\therefore A' = \frac{-vR}{uv'-u'v}, \quad B' = \frac{UR}{uv'-u'v}$$

$$\Rightarrow A = \int \frac{-vR}{uv'-u'v} dx + C_1$$

$$B = \int \frac{UR}{uv'-u'v} dx + C_2$$

Then the soln is given by

$$y = AU + BV$$

Note.

(4) & (5) are

$$A'U + B'v = 0$$

$$A'U' + B'v' = R$$

$$\therefore \frac{A'}{\begin{vmatrix} 0 & v \\ R & v' \end{vmatrix}} = \frac{B'}{\begin{vmatrix} U & 0 \\ u' & R \end{vmatrix}} = \frac{1}{\begin{vmatrix} U & v \\ u' & v' \end{vmatrix}}$$

Crammer's rule

$$\Rightarrow A' = \frac{-Rv}{\Delta}, \quad B' = \frac{UR}{\Delta}$$

etc.

Ex 1.3.1. Apply the method of variation of parameters to solve $\frac{dy}{dx} + ny = \sec nx$.

Soln. The eqn is $\frac{dy}{dx} + ny = \sec nx$.

Here, $P=0$, $Q=n^2$ and $R=\sec nx$.

$\therefore A.B.$ is $D^2 + n^2 = 0 \Rightarrow D = \pm in$.

$\therefore C.F. = C_1 \cos nx + C_2 \sin nx$.

Let $u = \cos nx$, $v = \sin nx$.

Let the mth be $y = Au + Bv$.

where $A'u + B'v = 0 \quad \dots \text{--- } 1$

Also $A'u' + B'v' = R \quad \dots \text{--- } 2$

$$\Rightarrow A = \int \frac{-vR}{uv' - vu'} dx + C_1$$

$$= \int \frac{-\sin nx \cdot \sec nx}{n \cdot \cos nx \cdot \sec nx - \sin nx(-\sec nx) n} dx + C_1$$

$$= -\frac{1}{n} \int \tan nx dx + C_1 \quad [uv' - vu' = n]$$

$$= -\frac{1}{n} \ln |\sec nx| \cdot \frac{1}{n} + C_1$$

$$= \frac{1}{n} \ln \sec nx + C_1.$$

$$\text{And } B = \int \frac{uR}{uv' - vu'} dx + C_2$$

$$= \int \frac{\cos nx \cdot \sec nx}{n} dx + C_2$$

$$= \int \frac{1}{n} dx + C_2 = \frac{1}{n} x + C_2$$

\therefore The soln is

$$y = Au + Bu$$

$$= \left(\frac{1}{n^2} \log \cos nx + C_1 \right) \cos nx + \left(\frac{x}{n} + C_2 \right) \sin nx.$$

$$= C_1 \cos nx + C_2 \sin nx + \frac{\cos nx}{n^2} \log \cos nx + \frac{x}{n} \sin nx.$$

Ex. 1.3.2

$$\frac{d^2y}{dx^2} + 4y = 4 \tan 2x$$

[G.U.]

Soln Here, $p=0, Q=4, R=4 \tan 2x$.

$$\text{A.E. is } D^2 + 4 = 0 \Rightarrow D = \pm 2i$$

$$\therefore CF = C_1 \cos 2x + C_2 \sin 2x$$

$$\therefore u = \cos 2x, v = \sin 2x.$$

$$\text{Let the soln be } y = Au + Bu.$$

$$\text{Then } A = \int \frac{-vR}{\Delta} dx + C_1$$

$$B = \int \frac{uR}{\Delta} dx + C_2$$

$$\text{And } \Delta = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2 \sin 2x & 2 \cos 2x \end{vmatrix}$$

$$= 2.$$

$$\therefore A = \int \frac{\sin 2x \cdot 4 \tan 2x}{2} dx + C_1$$

$$= -2 \int \frac{\sin^2 2x}{\cos 2x} dx + C_1$$

$$z = - \int (\sec x - \operatorname{cosec} x) dx + C_1$$

$$z = - \log(\sec 2x + \tan 2x) + \sin 2x + C_1$$

$$\beta = \int \frac{\sin 2x \cdot 4 \tan 2x}{2} dx + C_2$$

$$= \int 8 \sin 2x \cdot 2 dx + C_2$$

$$= - \cos 2x + C_2$$

\therefore The soln is $y = Au + Bu$

$$= \{ \sin 2x - \log(\sec 2x + \tan 2x) + C_1 \} \\ \cdot \cos 2x + (-\cos 2x + C_2) \sin 2x.$$

$$= C_1 \cos 2x + C_2 \sin 2x \\ - \cos 2x \log(\sec 2x + \tan 2x).$$

Ex. Solve $\frac{d^3y}{dx^3} + 9y = \sec 3x$.

Put $n=3$ in the above example.

(45)

$$\Rightarrow y = C_1 \cos 2x + C_2 \sin 2x - \cos 2x \log(\sec 2x + \tan 2x) + \text{constant}.$$

Sol $\frac{dy}{dx} + y = \cosec x.$

Ans $y = C_1 \cos x + C_2 \sin x - x \cos x + \sin x \log \sin x.$

Eo. $\frac{dy}{dx} + y = x.$ [Eqn with constant coefficient also].

Ans $y = C_1 \cos x + C_2 \sin x + x. //$